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THE $\mathrm{SL}_2(\mathbb{C})$ CASSON INVARIANT FOR SEIFERT FIBERED HOMOLOGY SPHERES AND SURGERIES ON TWIST KNOTS

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ABSTRACT

We derive a simple closed formula for the $SL_2(\mathbb{C})$ Casson invariant for Seifert fibered homology 3-spheres using the correspondence between $SL_2(\mathbb{C})$ character varieties and moduli spaces of parabolic Higgs bundles of rank two. These results are then used to deduce the invariant for Dehn surgeries on twist knots by combining computations of the Culler-Shalen norms with the surgery formula for the $SL_2(\mathbb{C})$ Casson invariant.

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1. Introduction

The 3-manifold invariant $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is defined in [11] by counting certain $SL_2(\mathbb{C})$ representations of $\pi_1\Sigma$ and can be regarded as the $SL_2(\mathbb{C})$ analogue of Casson's invariant. The goal of this paper is to present two techniques for computing $\lambda_{SL_2(\mathbb{C})}(\Sigma)$. The first is a direct approach and yields a simple closed formula for the $SL_2(\mathbb{C})$ Casson invariant for Seifert fibered homology spheres $\Sigma(a_1,\ldots,a_n)$. The second involves the surgery formula of [11] and requires calculation of the Culler-Shalen seminorms. This approach applies to give values of the invariant for 3-manifolds obtained by Dehn surgery along a twist knot.

We begin by introducing some notation and recalling the definition of the invariant $\lambda_{SL_2(\mathbb{C})}(\Sigma)$.

Given a finitely generated group π, denote by $R(\pi)$ the space of representations $\rho\colon \pi \to SL_2(\mathbb{C})$ and by $R^*(\pi)$ the subspace of irreducible representations. Recall from [9] that $R(\pi)$ has the structure of a complex affine algebraic set. The *character* of a representation ρ is the function $\chi_\rho\colon \pi \to \mathbb{C}$ defined by setting $\chi_\rho(\gamma) = tr(\rho(\gamma))$

for $\gamma \in \pi_1\Sigma$. The set of characters of $SL_2(\mathbb{C})$ representations is denoted $X(\pi)$ and also admits the structure of a complex affine algebraic set. Furthermore, there is a canonical projection $t\colon R(\pi) \to X(\pi)$ defined by $t\colon \rho \mapsto \chi_\rho$ which is surjective. Let $X^*(\pi)$ be the subspace of characters of irreducible representations. Given a manifold Σ, we denote by $R(\Sigma)$ the variety of $SL_2(\mathbb{C})$ representations of $\pi_1\Sigma$ and by $X(\Sigma)$ the associated character variety.

Suppose now Σ is a closed, orientable 3-manifold with a Heegaard splitting (W_1, W_2, F). Here, F is a closed orientable surface embedded in Σ, and W_1 and W_2 are handlebodies with boundaries $\partial W_1 = F = \partial W_2$ such that $\Sigma = W_1 \cup_F W_2$. The inclusion maps $F \hookrightarrow W_i$ and $W_i \hookrightarrow \Sigma$ induce a diagram of surjections

$$\begin{tikzcd} & \pi_1 W_1 \arrow[dr] & \\ \pi_1 F \arrow[ur] \arrow[dr] & & \pi_1\Sigma. \\ & \pi_1 W_2 \arrow[ur] & \end{tikzcd}$$

This diagram induces a diagram of the associated representation varieties, where all arrows are reversed and are injective rather than surjective. On the level of character varieties, this gives a diagram

$$\begin{tikzcd} & X(W_1) \arrow[dl] & \\ X(F) & & X(\Sigma), \arrow[ul] \arrow[dl] \\ & X(W_2) \arrow[ul] & \end{tikzcd}$$

of injections. This identifies $X(\Sigma)$ as the intersection

$$X(\Sigma) = X(W_1) \cap X(W_2) \subset X(F).$$

There are natural orientations on all the character varieties determined by their complex structures. The invariant $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is defined as an oriented intersection number of $X^*(W_1)$ and $X^*(W_2)$ in $X^*(F)$ which counts only compact, zero-dimensional components of the intersection. Specifically, there exist a compact neighborhood U of the zero-dimensional components of $X^*(W_1) \cap X^*(W_2)$ which is disjoint from the higher dimensional components of the intersection and an isotopy $h\colon X^*(F) \to X^*(F)$ supported in U such that $h(X^*(W_1))$ and $X^*(W_2)$ intersect transversely in U. Then given a zero-dimensional component χ of the intersection $h(X^*(W_1)) \cap X^*(W_2)$, we may set $\varepsilon_\chi = \pm 1$, depending on whether the orientation of $h(X^*(W_1))$ followed by that of $X^*(W_2)$ agrees with or disagrees with the orientation of $X^*(F)$ at χ.

Definition 1.1. *Let $\lambda_{SL_2(\mathbb{C})}(\Sigma) = \sum_\chi \varepsilon_\chi$, where the sum is over all zero-dimensional components χ of the intersection $h(X^*(W_1)) \cap X^*(W_2)$.*

The next result recalls from [11] the basic properties of the $SL_2(\mathbb{C})$ Casson invariant. Of these, properties (i)–(iv) are proved explicitly in [11] and (v) is implicit in the definition of $\lambda_{SL_2(\mathbb{C})}(\Sigma)$. Property (vi), additivity under connected sum for $\mathbb{Z}_2$ homology spheres, is a consequence of Theorem 3.1.

Theorem 1.2. *The invariant $\lambda_{SL_2(\mathbb{C})}$ satisfies the following properties:*

(i) *For any 3-manifold Σ, $\lambda_{SL_2(\mathbb{C})}(\Sigma) \geq 0$.*
(ii) *If Σ is hyperbolic, then $\lambda_{SL_2(\mathbb{C})}(\Sigma) > 0$.*
(iii) *If $\lambda_{SL_2(\mathbb{C})}(\Sigma) > 0$, then there exists an irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$. In particular, if $\pi_1\Sigma$ is abelian, then $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$.*
(iv) *$\lambda_{SL_2(\mathbb{C})}$ satisfies a surgery formula. (See Theorem 4.7 for details.)*
(v) *$\lambda_{SL_2(\mathbb{C})}(-\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma)$, where $-\Sigma$ is Σ with the opposite orientation.*
(vi) *$\lambda_{SL_2(\mathbb{C})}(\Sigma_1 \# \Sigma_2) = \lambda_{SL_2(\mathbb{C})}(\Sigma_1) + \lambda_{SL_2(\mathbb{C})}(\Sigma_2)$ for $\mathbb{Z}_2$ homology 3-spheres Σ_1, Σ_2.*

There is an alternative formulation of $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ in terms of the intersection theory of algebraic varieties [14]. For an isolated point χ in $X^*(W_1) \cap X^*(W_2)$, set m_χ equal to the intersection multiplicity of χ in the intersection cycle $X^*(W_1) \cdot X^*(W_2)$. (See [11] for details.) Then

$$\lambda_{SL_2(\mathbb{C})}(\Sigma) = \sum_{\chi} m_\chi, \tag{1.1}$$

where the sum is over all isolated points χ in $X^*(W_1) \cap X^*(W_2)$.

We will use equation (1.1) to compute $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ in several interesting cases where the intersection multiplicities m_χ are not too difficult to determine. In Section 2, we give a simple criterion under which the components of $X^*(W_1) \cap X^*(W_2)$ are all zero-dimensional with intersection multiplicity one. In this fortuitous case, one can compute $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ directly from the character variety $X(\Sigma)$ without reference to the Heegaard splitting or the isotopy h. After verifying that this criterion holds for Brieskorn spheres, we use it to calculate $\lambda_{SL_2(\mathbb{C})}(\Sigma)$, first for Brieskorn spheres, then more generally for Seifert fibered homology spheres. In Section 3 we state and prove a formula for $\lambda_{SL_2(\mathbb{C})}$ for connected sums of rational homology 3-spheres. In Section 4 we turn our attention to 3-manifolds resulting from Dehn surgery on a knot in a homology 3-sphere Σ and recall from [11] the surgery formula for the $SL_2(\mathbb{C})$ Casson invariant. Finally in Section 5 we combine the calculations of Section 2 with the surgery formula and deduce a formula for the $SL_2(\mathbb{C})$ Casson invariant for 3-manifolds obtained by Dehn surgery on a twist knot.

2. Seifert fibered homology spheres

Given a representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$, we denote by $H^*(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho})$ the cohomology groups of Σ with coefficients in $\mathfrak{sl}_2(\mathbb{C})$ twisted by

$$Ad\,\rho\colon \pi_1\Sigma \to \operatorname{Aut}(\mathfrak{sl}_2(\mathbb{C})).$$

Theorem 2.1. *If Σ is a closed oriented 3-manifold and $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ is an irreducible representation with $H^1(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$, then its character χ_ρ is an isolated point of the intersection $X^*(W_1) \cap X^*(W_2)$ with intersection multiplicity 1.*

Proof. Let $\rho_0\colon \pi_1 F \longrightarrow \pi_1\Sigma \xrightarrow{\rho} SL_2(\mathbb{C})$ be the representation on the Heegaard surface F obtained by pullback. Similarly, for $i = 1,2$ let $\rho_i\colon \pi_1 W_i \longrightarrow \pi_1\Sigma \xrightarrow{\rho} SL_2(\mathbb{C})$ be the representation on the handlebody W_i. Because ρ is irreducible and because each of the three inclusion maps induces a surjection on the level of fundamental groups, the three representations ρ_0, ρ_1, and ρ_2 are irreducible as well.

The Zariski tangent space to $\chi_\rho \in X(\Sigma)$ is a subspace of $H^1(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho})$. Mayer-Vietoris identifies this cohomology group with the intersection of the images of $H^1(W_1, \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho_1})$ and $H^1(W_2, \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho_2})$ in $H^1(F, \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho_0})$. The condition that $H^1(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$ therefore guarantees that $X(W_1)$ and $X(W_2)$ intersect transversely at χ_ρ, which is therefore isolated. Since all spaces are oriented as complex varieties and the intersection is transverse, this shows that χ_ρ contributes $+1$ to $\lambda_{SL_2(\mathbb{C})}(\Sigma)$. □

Corollary 2.2. *If Σ is a closed oriented 3-manifold such that $H^1(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$ for every irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$, then $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is an exact count of the conjugacy classes of irreducible representations of $\pi_1\Sigma$ in $SL_2(\mathbb{C})$ – i.e. $\lambda_{SL_2(\mathbb{C})}(\Sigma) = |X^*(\Sigma)|$.*

We now apply these results to compute $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ for Seifert fibered homology spheres. We begin with the Brieskorn manifolds

$$\Sigma(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\} \cap S^5.$$

If p, q, and r are positive and pairwise relatively prime, then $\Sigma(p,q,r)$ is an integral homology sphere, called a Brieskorn sphere.

Theorem 2.3. *If $\Sigma(p,q,r)$ is a Brieskorn sphere and $\rho\colon \pi_1\Sigma(p,q,r) \to SL_2(\mathbb{C})$ is irreducible, then $H^1(\Sigma(p,q,r), \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$. Furthermore,*

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,r)) = \frac{(p-1)(q-1)(r-1)}{4}.$$

Proof. One way to prove this is to use the correspondence between the character variety $X(\Sigma)$ and certain moduli spaces of parabolic Higgs bundles over $\mathbb{CP}^1$. This is the approach we will adopt in our proof of Theorem 2.7, but here we give an independent and elementary argument.

The fundamental group of $\Sigma(p,q,r)$ has a presentation

$$\pi_1\Sigma(p,q,r) = \langle x, y, z, h \mid h \text{ is central}, \ x^p = h^a, y^q = h^b, z^r = h^c, xyz = 1\rangle, \quad (2.1)$$

for integers a, b, c satisfying

$$aqr + bpr + cpq = 1. \quad (2.2)$$

Lemma 2.4. $H^1(\Sigma(p,q,r); \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$ *for all* $\rho\colon \pi_1\Sigma(p,q,r) \to SL_2(\mathbb{C})$.

Proof. Since $\Sigma(p,q,r)$ is a homology sphere, the cohomology vanishes when ρ is the trivial representation. Thus, we assume ρ is irreducible, and therefore $\rho(h) = \pm I$. Hence

$$\rho(x)^{2p} = \rho(y)^{2q} = \rho(z)^{2r} = \rho(h)^2 = I,$$

so the eigenvalues of $\rho(x), \rho(y)$, and $\rho(z)$ are $2p$-th, $2q$-th, and $2r$-th roots of 1, respectively. Moreover, since ρ is irreducible, none of these eigenvalues is ± 1.

To simplify notation, we write π for $\pi_1\Sigma(p,q,r)$. We use the Fox calculus to determine the dimension of the space of 1-cocycles $Z^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho})$. The 1-cocycles are determined by elements $X, Y, Z, H \in \mathfrak{sl}_2(\mathbb{C})$ satisfying the equations obtained by taking Fox derivatives of the relations in (2.1). The commutation relation $hxh^{-1}x^{-1}$ gives

$$H + Ad\,\rho(h)X - Ad\,\rho(x)H - X = 0.$$

Since $Ad\,\rho(h)X = X$, we see that H lies in the kernel of $1 - Ad\,\rho(x)$. Similarly, H lies in the kernels of $1 - Ad\,\rho(y)$ and $1 - Ad\,\rho(z)$. Since ρ is irreducible with image generated by $\rho(x), \rho(y)$ and $\rho(z)$, this implies that $H = 0$.

Setting $H = 0$ in the remaining equations, we obtain:

$$\begin{aligned}
(1 + Ad\,\rho(x) + \cdots + Ad\,\rho(x^{p-1}))X &= 0, && (2.3)\\
(1 + Ad\,\rho(y) + \cdots + Ad\,\rho(y^{q-1}))Y &= 0, && (2.4)\\
(1 + Ad\,\rho(z) + \cdots + Ad\,\rho(z^{r-1}))Z &= 0, && (2.5)\\
X + Ad\,\rho(x)Y + Ad\,\rho(xy)Z &= 0. && (2.6)
\end{aligned}$$

By (2.3), X lies in the kernel of $(1 + Ad\,\rho(x) + \cdots + Ad\,\rho(x^{p-1}))$. Now since the eigenvalues of $\rho(x)$ are distinct $2p$-th roots of 1, we see that there is an isomorphism $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$ such that $Ad\,\rho(x)$ is given in the new coordinates by

$$Ad\,\rho(x)(\zeta_1, \zeta_2, \zeta_3) = (e^{\pi ik/p}\zeta_1, e^{-\pi ik/p}\zeta_2, \zeta_3)$$

for some $1 \leq k < p$. Then clearly the kernel of $(1 + Ad\,\rho(x) + \cdots + Ad\,\rho(x^{p-1}))$ is the 2-dimensional subspace of $\mathfrak{sl}_2(\mathbb{C})$ corresponding to the subspace $\zeta_3 = 0$ in $\mathbb{C}^3$. Similarly, equations (2.4) and (2.5) imply that the kernels of $(1 + Ad\,\rho(y) + \cdots + Ad\,\rho(y^{p-1}))$ and $(1 + Ad\,\rho(z) + \cdots + Ad\,\rho(z^{p-1}))$ are 2-dimensional. Finally, since ρ is irreducible, we see that (2.6) imposes three independent conditions. We conclude that

$$\dim Z^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 2 + 2 + 2 - 3 = 3.$$

Now the irreducibility of ρ also implies that $H^0(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$. Hence the space of coboundaries has $\dim B^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 3$, and it follows that

$$\dim H^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = \dim Z^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) - \dim B^1(\pi; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0. \quad \square$$

Now Corollary 2.2 applies to show that $\lambda_{SL_2(\mathbb{C})}(\Sigma) = |X^*(\Sigma)|$, and we turn to the problem of enumerating the characters in $X^*(\Sigma)$. Our enumeration will be given in terms of the numbers of distinct conjugacy classes of appropriate non-central roots of unity in $SL_2(\mathbb{C})$.

Reordering p, q, and r as necessary, we may assume that q and r are odd. We establish a one-to-one correspondence between characters of irreducible representations $\rho\colon \pi_1\Sigma(p,q,r) \to SL_2(\mathbb{C})$ and characters of irreducible representations $\bar{\rho}\colon T(2p,q,r) \to SL_2(\mathbb{C})$, where $T(2p,q,r)$ is the triangle group

$$T(2p,q,r) = \langle x, y, z \mid x^{2p} = y^q = z^r = xyz = 1\rangle.$$

Assume first that $\rho\colon \pi_1\Sigma(p,q,r) \to SL_2(\mathbb{C})$ is given and irreducible. If b is odd, we may replace it with $b+q$ and a with $a-p$ so that b is even. Likewise, we may replace c with $c+r$ and a with $a-p$ as necessary to guarantee that c is even.

Since b is even, $\rho(y)$ must be a q-th root of $\rho(h)^b = I$. Similarly, since c is even, $\rho(z)$ must be an r-th root of I. Now equation (2.2) implies a is odd. Hence $\rho(x)$ is a p-th root of $\rho(h)^a = \rho(h) = \pm I$, and therefore $\rho(x)$ is a $2p$-th root of I. Moreover $\rho(xyz) = I$. Thus ρ determines an irreducible representation $\bar{\rho}\colon T(2p,q,r) \to SL_2(\mathbb{C})$ given by $\bar{\rho}(x) = \rho(x), \bar{\rho}(y) = \rho(y)$, and $\bar{\rho}(z) = \rho(z)$.

Now assume $\bar{\rho}\colon T(2p,q,r) \to SL_2(\mathbb{C})$ is given and irreducible. Setting $\rho(h) = \bar{\rho}(x^p) = \pm I$ defines an irreducible representation $\rho\colon \pi_1\Sigma(p,q,r) \to SL_2(\mathbb{C})$ in the obvious way.

Thus, $\lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,r)) = |X^*(\Sigma(p,q,r))| = |X^*(T(2p,q,r))|$. We now enumerate the characters in $|X^*(T(2p,q,r))|$. Recall that two irreducible representations $\rho_1, \rho_2 : G \to SL_2(\mathbb{C})$ have the same character if and only if they are conjugate. Thus, we enumerate the conjugacy classes of irreducible representations $\rho\colon T(2p,q,r) \to SL_2(\mathbb{C})$.

Note that $T(2p,q,r)$ is generated by the two elements x and y, since $xyz = 1$. But for any group G generated by two elements g_1 and g_2, an irreducible representation $\phi\colon G \to SL_2(\mathbb{C})$ is determined up to conjugacy by the traces of $\phi(g_1), \phi(g_2)$, and $\phi(g_1g_2)$. To see this, recall that the Cayley-Hamilton theorem implies the trace relation:

$$tr(AB) - tr(A)\, tr(B) + tr(A^{-1}B) = 0$$

for all $A, B \in SL_2(\mathbb{C})$. Arguing by induction, this relation implies that the trace of any word in A and B is determined by $tr(A), tr(B)$, and $tr(AB)$.

To enumerate the conjugacy classes of irreducible representations $\rho\colon T(2p,q,r) \to SL_2(\mathbb{C})$, we need only determine the possible traces of $\rho(x), \rho(y)$, and $\rho(xy) = \rho(z^{-1})$. Since these matrices have order $2p, q$, and r, respectively, in $SL_2(\mathbb{C})$, it is clear that the eigenvalues of these matrices must be $2p$-th, q-th, and r-th roots of unity. Moreover, since ρ is irreducible, none of the eigenvalues are ± 1. The next lemma is used to identify the eigenvalues associated to irreducible representations $\rho\colon T(2p,q,r) \to SL_2(\mathbb{C})$.

Lemma 2.5. *Suppose $A, B, C \in SL_2(\mathbb{C})$ satisfy $tr\, A = 2\cos\alpha, tr\, B = 2\cos\beta$, and $tr\, C = 2\cos\gamma$ and for $\alpha, \beta, \gamma \in (0, \pi)$. Then there exists $P \in SL_2(\mathbb{C})$ with $tr(APBP^{-1}) = tr\, C$.*

Remark 2.6. *It is interesting to compare this result to the $SU(2)$ analogue, where necessary and sufficient conditions for the existence of an $SU(2)$ representation with specified traces are given by the quantum Clebsch-Gordon coefficients.*

Proof. The trace conditions guarantee that A, B, and C are conjugate to unitary matrices and as such are diagonalizable; thus we may assume that

$$A = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}.$$

We will actually find $P \in SL_2(\mathbb{R})$ that satisfies the conclusion by writing

$$P = \begin{pmatrix} u & v \\ -1 & 1 \end{pmatrix},$$

and computing

$$\begin{aligned} tr(APBP^{-1}) &= e^{i(\alpha+\beta)}u + e^{i(\alpha-\beta)}v + e^{i(\beta-\alpha)}v + e^{-i(\alpha+\beta)}u \\ &= 2u\cos(\alpha+\beta) + 2v\cos(\alpha-\beta). \end{aligned}$$

To prove the lemma, we need to find u and v satisfying

$$\begin{aligned} 1 &= u + v \\ \cos\gamma &= u\cos(\alpha+\beta) + v\cos(\alpha-\beta). \end{aligned}$$

Basic linear algebra shows that these equations can be solved unless $\cos(\alpha+\beta) = \cos(\alpha-\beta)$, which is equivalent to the condition that $\sin\alpha = 0$ or $\sin\beta = 0$. Since $\alpha, \beta \in (0, \pi)$, we see that $\sin\alpha \neq 0 \neq \sin\beta$. $\square$

Thus, given any 3 elements $\alpha, \beta, \gamma \in (0, \pi)$ such that $e^{i\alpha}$ is a $2p$-th root of unity, $e^{i\beta}$ is a q-th root of unity, and $e^{i\gamma}$ is an r-th root of unity, we may define an irreducible representation $\rho\colon T(2p, q, r) \to SL_2(\mathbb{C})$ with $tr\rho(x) = 2\cos 2\alpha, tr\rho(y) = 2\cos\beta$, and $tr\rho(xy) = 2\cos\gamma$ by setting

$$\rho(x) = \begin{pmatrix} e^{2i\alpha} & 0 \\ 0 & e^{-2i\alpha} \end{pmatrix} \quad \text{and} \quad \rho(y) = P\begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}P^{-1},$$

where P is the matrix found in Lemma 2.5.

Finally, noting that there are $p-1$ $2p$-th roots of unity, $\frac{q-1}{2}$ q-th roots of unity, and $\frac{r-1}{2}$ r-th roots of unity in the open semicircle $\{e^{it} \mid 0 < t < \pi\}$, we find that

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(p, q, r)) = |X^*(T(2p, q, r))| = \tfrac{1}{4}(p-1)(q-1)(r-1). \quad \square$$

Theorem 2.7. *Suppose $a_1, \ldots, a_n$ are positive integers that are pairwise relatively prime, and denote by $\Sigma(a_1, \ldots, a_n)$ the associated Seifert fibered homology sphere. Then*

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(a_1, \ldots, a_n)) = \sum_{1 \le i_1 < i_2 < i_3 \le n} \frac{(a_{i_1}-1)(a_{i_2}-1)(a_{i_3}-1)}{4}.$$

Remark 2.8. *The right-hand side of this formula equals $\frac{1}{4}\sigma_3\,(a_1-1, \ldots, a_n-1)$, the elementary symmetric polynomial of degree 3 in the n variables $a_1-1, \ldots, a_n-1$.*

Proof. We begin with the presentation

$$\pi_1\Sigma(a_1,\ldots,a_n)=\langle x_1,\ldots,x_n,h \mid h \text{ central},\ x_i^{a_i}=h^{-b_i}, x_1\cdots x_n=1\rangle. \tag{2.7}$$

Here, the b_i are not unique but must satisfy

$$\sum_{i=1}^n b_i\, a_1\cdots\widehat{a}_i\cdots a_n=1, \tag{2.8}$$

where $\widehat{a}_i$ indicates this term is omitted.

By reordering, we may assume that a_i is odd for $i>1$. There is a correspondence between the character variety $X^*(\Sigma(a_1,\ldots,a_n))$ and that of the n-gon group

$$T(2a_1,a_2,\ldots,a_n)=\langle x_1,\ldots,x_n \mid x_1^{2a_1}=x_2^{a_2}=\cdots=x_n^{a_n}=x_1\cdots x_n=1\rangle.$$

This correspondence may be established as in the proof of Theorem 2.3 for the case $n=3$.

Now Simpson's theorem [20] establishes a correspondence between connected components of the character variety $X^*(T(2a_1,a_2,\ldots,a_n))$ and moduli spaces of parabolic Higgs bundles. Components of $X^*(T(2a_1,a_2,\ldots,a_n))$ are indexed by the possible traces of the images of the generators $x_1,\ldots,x_n$. Since each of them has finite order, any irreducible representation must map each x_i to a unitary matrix with eigenvalues given by roots of unity of the appropriate order. Thus, the traces of the images of $x_1,x_2,\ldots,x_n$ are real numbers of the form

$$2\cos(2\pi\alpha_1), 2\cos(2\pi\alpha_2),\ldots,2\cos(2\pi\alpha_n),$$

where $\alpha_1=k_1/2a_1$ and $\alpha_i=k_i/a_i$ for $i>1$, and where $k_i\in\mathbb{Z}$ satisfies $0\le k_1\le a_1$ for $i=1$ and $0\le k_i<a_i/2$ for $i>1$.

We set $\alpha=(\alpha_1,\ldots,\alpha_n)$ and let $\mathscr{M}_\alpha$ denote the moduli space of rank two parabolic Higgs bundles of parabolic degree zero over $\mathbb{CP}^1$ with n marked points $p_1,\ldots,p_n$ and weights $\alpha_i, 1-\alpha_i$ at p_i. Given this choice of weights, one can easily verify that every semistable parabolic Higgs bundle is actually stable. As a result, the moduli space $\mathscr{M}_\alpha$ is smooth of complex dimension $2m-6$, where

$$m=m(\alpha)=\left|\left\{\alpha_i \mid \alpha_i\in\left(0,\tfrac12\right)\right\}\right|,$$

the number of nontrivial flags in the quasiparabolic structure. (If $m<3$, then $\mathscr{M}_\alpha=\varnothing$. This corresponds to the requirement that an irreducible representation of $T(2a_1,a_2,\ldots,a_n)$ must send at least three generators to noncentral elements in $SL_2(\mathbb{C})$.)

Thus, the zero-dimensional components of $X^*(T(2a_1,a_2,\ldots,a_n))$ are in one-to-one correspondence with the subset $\{\alpha \mid m(\alpha)=3\}$ of all possible weights. Such a weight $\alpha=(\alpha_1,\ldots,\alpha_n)$ is obtained by choosing $1\le i_1<i_2<i_3\le n$ with $\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}\in\left(0,\frac12\right)$ and all remaining α_j equal to 0 or $\frac12$. (Note: only α_1 is allowed to equal $\frac12$ here, since for $\chi\in X^*(T(2a_1,a_2,\ldots,a_n))$, the only generator whose trace may equal -2 is x_1. Note also that as in the proof of Theorem 2.3, we may apply Lemma 2.5 to show that there exists an irreducible representation of $T(2a_1,a_2,\ldots,a_n)$ with the given weights.)

A straightforward generalization of Lemma 2.4 shows that such components correspond to isolated components of $X^*(\Sigma(a_1,\ldots,a_n))$ of intersection multiplicity one, and so our task is simply to enumerate them. To do this, consider the cases $i_1 = 1$ and $i_1 > 1$ separately.

In the case $i_1 = 1$, we have

$$\alpha_1 \in \left\{ \tfrac{1}{2a_1}, \ldots, \tfrac{a_1-1}{2a_1} \right\},$$

a set with $a_1 - 1$ elements. (This enumeration corresponds to the one used previously in counting conjugacy classes of roots of unity in $SL_2(\mathbb{C})$. In particular, there are $a_1 - 1$ distinct conjugacy classes of noncentral $2a_1$-th roots of unity in $SL_2(\mathbb{C})$.) Likewise,

$$\alpha_{i_2} \in \left\{ \tfrac{1}{a_{i_2}}, \ldots, \tfrac{a_{i_2}-1}{2a_{i_2}} \right\} \quad \text{and} \quad \alpha_{i_3} \in \left\{ \tfrac{1}{a_{i_3}}, \ldots, \tfrac{a_{i_3}-1}{2a_{i_3}} \right\},$$

which are sets with $\frac{1}{2}(a_2 - 1)$ and $\frac{1}{2}(a_3 - 1)$ elements, respectively. (Note that a_{i_2} and a_{i_3} are both odd, and that $\frac{1}{2}(a_2 - 1)$ and $\frac{1}{2}(a_3 - 1)$ are precisely the number of distinct conjugacy classes of noncentral a_{i_2}-th and a_{i_3}-th roots of unity in $SL_2(\mathbb{C})$, respectively.) Since $m(\alpha) = 3$, we take all other $a_j = 0$, so this case give a total of $\frac{1}{4}(a_{i_1} - 1)(a_{i_2} - 1)(a_{i_3} - 1)$ isolated point components.

In the case $i_1 > 1$, one counts as above to see that for $k = 1, 2, 3$,

$$\alpha_{i_k} \in \left\{ \tfrac{1}{a_{i_k}}, \ldots, \tfrac{a_{i_k}-1}{2a_{i_k}} \right\},$$

a set with $\frac{1}{2}(a_{i_k} - 1)$ elements. In this case, α_1 may equal 0 or $\frac{1}{2}$, which gives an extra factor of 2, and all other $\alpha_j = 0$. Thus, there are $\frac{1}{4}(a_{i_1} - 1)(a_{i_2} - 1)(a_{i_3} - 1)$ isolated point components. The proof of the theorem is completed by summing over all possible $1 \leq i_1 < i_2 < i_3 \leq n$. □

For Brieskorn spheres, Fintushel and Stern showed that Casson's invariant satisfies

$$\lambda_{SU(2)}(\Sigma(p,q,r) = \tfrac{1}{8} \text{signature}\, M(p,q,r),$$

where $M(p,q,r)$ denotes the Milnor fiber of the singularity [13]. This equation was shown to hold more generally by Neumann and Wahl, who have conjectured that $\lambda_{SU(2)}(\Sigma) = \frac{1}{8}\text{signature}\, M$ for any link of a normal complete intersection singularity [17].

Theorem 2.3 shows that the $SL_2(\mathbb{C})$ Casson invariant satisfies

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,r)) = \tfrac{1}{4}\mu(p,q,r), \tag{2.9}$$

where $\mu(p,q,r)$ denotes the Milnor number [16]. Theorem 2.7 shows this formula does not extend to Seifert fibered homology spheres $\Sigma(a_1,\ldots,a_n)$ with $n \geq 4$. We explain briefly the correct generalization of (2.9).

Consider the components $X_j \subset X^*(\Sigma(a_1,\ldots,a_n))$ of highest dimension, namely those with $\dim X_j = 2n - 6$. By the formula on p.597 of [1], we have $\chi(X_j) = (n-1)(n-2)2^{n-4}$ for each component X_j. Moreover, a lattice point count similar to the argument given in the proofs of Theorems 2.3 and 2.7 shows that there are

$\frac{1}{4}(a_1-1)\dots(a_n-1)$ such components. Finally, by Theorem 9.1 of [16] we have $\mu(a_1,\dots,a_n)=(a_1-1)\dots(a_n-1)$. It follows that

$$\sum_j \chi(X_j)=(n-1)(n-2)2^{n-6}\mu(a_1,\dots,a_n).$$

3. Connected sum formula

Theorem 3.1. *If Σ_1 and Σ_2 are rational homology spheres, then*

$$\lambda_{SL_2(\mathbb{C})}(\Sigma_1\#\Sigma_2)=|H_1(\Sigma_2;\mathbb{Z}_2)|\cdot\lambda_{SL_2(\mathbb{C})}(\Sigma_1)+|H_1(\Sigma_1;\mathbb{Z}_2)|\cdot\lambda_{SL_2(\mathbb{C})}(\Sigma_2),$$

As a consequence, the invariant $\lambda_{SL_2(\mathbb{C})}$ is additive under connected sum for $\mathbb{Z}_2$-homology spheres.

Proof. Since $\pi_1(\Sigma_1\#\Sigma_2)=\pi_1(\Sigma_1)*\pi_1(\Sigma_2)$, we have $R(\Sigma_1\#\Sigma_2)=R(\Sigma_1)\times R(\Sigma_2)$. Suppose ρ_1,ρ_2 are $SL_2(\mathbb{C})$ representations on Σ_1,Σ_2, with stabilizer groups Γ_1,Γ_2, respectively. Then under the conjugation action, the orbit of ρ_1 in $R(\Sigma_1)$ is $SL_2(\mathbb{C})/\Gamma_1$ and the orbit of ρ_2 in $R(\Sigma_2)$ is $SL_2(\mathbb{C})/\Gamma_2$. We are interested only in $SL_2(\mathbb{C})$ representations $\rho=(\rho_1,\rho_2)\in R(\Sigma_1\#\Sigma_2)$ which are irreducible and map to isolated points under $t\colon R(\Sigma_1\#\Sigma_2)\to X(\Sigma_1\#\Sigma_2)$.

Given a representation $\rho=(\rho_1,\rho_2)\in R(\Sigma_1\#\Sigma_2)$ and an element $g\in SL_2(\mathbb{C})$, we may define a representation $\rho_g\in R(\Sigma_1\#\Sigma_2)$ by $\rho_g=(g\cdot\rho_1,\rho_2)$. In general, ρ_g lies in a different conjugacy class from ρ. In fact, the conjugacy class $[\rho]$ lies on a family $[\rho_g]$ of conjugacy classes parameterized by the double coset $\Gamma_1\backslash SL_2(\mathbb{C})/\Gamma_2$, and the conjugacy classes of irreducible representations in this family are in one-to-one correspondence with characters. In particular, if an irreducible representation $\rho=(\rho_1,\rho_2)$ maps to an isolated point in $X(\Sigma_1\#\Sigma_2)$, then either Γ_1 or Γ_2 must equal $SL_2(\mathbb{C})$. This is the case precisely when either ρ_1 or ρ_2 is central.

Suppose then that ρ_2 is central, so the image of ρ_2 is in $\{\pm I\}$. It follows easily that $\rho=(\rho_1,\rho_2)$ is irreducible if and only if ρ_1 is irreducible. Further, the resulting character χ_ρ will be isolated in $X^*(\Sigma_1\#\Sigma_2)$ if and only if χ_{ρ_1} is isolated in $X^*(\Sigma_1)$. (Here, we use the Mayer-Vietoris theorem and the fact that $H^1(\Sigma_2;\mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho_2})=0$, which follows since ρ_2 is central and Σ_2 is a rational homology 3-sphere.) One can piece together a compactly supported perturbation from Σ_1 with the trivial perturbation on Σ_2 to make the intersection transverse in a neighborhood of χ_ρ, showing in essence that $m_{\chi_\rho}=m_{\chi_{\rho_1}}$.

An analogous argument shows that if ρ_1 is central, then $\rho=(\rho_1,\rho_2)$ is irreducible and isolated if and only if ρ_2 is irreducible and isolated, and if so then $m_{\chi_\rho}=m_{\chi_{\rho_2}}$. The theorem follows from the additional observation that the central representations of $\pi_1\Sigma_i$ are in one-to-one correspondence with elements of $H_1(\Sigma_i;\mathbb{Z}_2)$. □

4. Dehn surgery formula for small knots

Theorem 4.8 of [11] gives a formula for the $SL_2(\mathbb{C})$ Casson invariant for Dehn surgeries on small knots in homology spheres in terms of a weighted sum of Culler-Shalen seminorms. In this section, we restate that formula, incorporating the comments from [12]. We begin by reviewing the notation of [9, 8].

Suppose M is a compact, irreducible, orientable 3-manifold with boundary a torus. An *incompressible surface* in M is a properly embedded surface $(F, \partial F) \to (M, \partial M)$ such that $\pi_1 F \to \pi_1 M$ is injective and no component of F is a 2-sphere bounding a 3-ball. An *essential surface* in M is an incompressible surface in M, no component of which is boundary parallel. The manifold M is called *small* if it does not contain a closed essential surface, and a knot K in Σ is called *small* if its complement $\Sigma \setminus \tau(K)$ is a small manifold.

If γ is a simple closed curve in ∂M, the Dehn filling of M along γ will be denoted by $M(\gamma)$; it is the closed 3-manifold obtained by identifying a solid torus with M along their boundaries so that γ bounds a disk. Note that the homeomorphism type of $M(\gamma)$ depends only on the *slope* of γ – that is, the unoriented isotopy class of γ. Primitive elements in $H_1(\partial M;\mathbb{Z})$ determine slopes under a two-to-one correspondence.

If F is an essential surface in M with nonempty boundary, then all of its boundary components are parallel and the slope of one (hence all) of these curves is called the *boundary slope* of F. A slope is called a *strict boundary slope* if it is the boundary slope of an essential surface that is not the fiber of any fibration of M over S^1.

For each $\gamma \in \pi_1 M$, there is a regular map $I_\gamma \colon X(M) \to \mathbb{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$. Let $e\colon H_1(\partial M;\mathbb{Z}) \to \pi_1(\partial M)$ be the inverse of the Hurewicz isomorphism. Identifying $e(\alpha) \in \pi_1(\partial M)$ with its image in $\pi_1 M$ under the natural map $\pi_1(\partial M) \to \pi_1 M$, we obtain a well-defined function $I_{e(\alpha)}$ on $X(M)$ for each $\alpha \in H_1(\partial M;\mathbb{Z})$. Let $f_\alpha \colon X(M) \to \mathbb{C}$ be the regular function defined by $f_\alpha = I_{e(\alpha)} - 2$ for $\alpha \in H_1(\partial M;\mathbb{Z})$.

Remark 4.1. *Our function f_α does not agree with that of the papers* [9, 10, 4, 5], *where f_α denotes $I^2_{e(\alpha)} - 4$.*

Let $r\colon X(M) \to X(\partial M)$ be the restriction map induced by $\pi_1(\partial M) \to \pi_1 M$. Suppose X_i is a one-dimensional component of $X(M)$ such that $r(X_i)$ is one-dimensional. Let $f_{i,\alpha}\colon X_i \to \mathbb{C}$ denote the regular function obtained by restricting f_α to X_i for each i.

Let $\widetilde{X}_i$ denote the smooth, projective curve birationally equivalent to X_i. Regular functions on X_i extend to rational functions on $\widetilde{X}_i$. We abuse notation and denote the extension of $f_{i,\alpha}$ to $\widetilde{X}_i$ by $f_{i,\alpha}\colon \widetilde{X}_i \to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

A generalization of the argument in Section 1.4 of [10] proves the following result.

Proposition 4.2. *Suppose X_i is a one-dimensional component of $X(M)$ which contains an irreducible character and whose restriction $r(X_i)$ is also one-dimensional.*

There exists a seminorm $\|\cdot\|_i$ on the real vector space $H_1(\partial M;\mathbb{R})$ satisfying

$$\|\alpha\|_i = \tfrac{1}{2}\deg(f_{i,\alpha})$$

for all α in the lattice $H_1(\partial M;\mathbb{Z})$.

Remark 4.3. *The relationship between $\|\cdot\|_i$ and the Culler-Shalen norm introduced in* [10] *is as follows: Suppose M is hyperbolic. Let X_0 be the unique component of $X(M)$ containing the character of the $SL_2(\mathbb{C})$ lift of the associated discrete, faithful representation $\rho\colon \pi_1 M \to PSL_2(\mathbb{C})$. Then Culler, Gordon, Luecke and Shalen proved the existence of a norm $\|\cdot\|_{CS}$ on $H_1(\partial M;\mathbb{R})$ such that $\|\alpha\|_{CS} = \deg\left(I^2_{0,e(\alpha)} - 4\right)$ for all $\alpha \in H_1(\partial M;\mathbb{Z})$. (See Section 1.4 of* [10]*.) It follows immediately from the definitions that $\|\cdot\|_{CS} = 4\|\cdot\|_0$. A similar relationship holds between $\|\cdot\|_i$ and the Culler-Shalen seminorms described in* [4, 5] *associated to the remaining curves X_i.*

We will relate the $SL_2(\mathbb{C})$-Casson invariant of a manifold obtained by Dehn filling to this seminorm; however we must impose certain restrictions on the filling slope.

Definition 4.4. *The slope of a simple closed curve γ in ∂M is called* irregular *if there exists an irreducible representation $\rho\colon \pi_1 M \to SL_2(\mathbb{C})$ such that*

(i) *the character χ_ρ of ρ lies on a one-dimensional component X_i of $X(M)$ such that $r(X_i)$ is one-dimensional,*
(ii) *$tr\,\rho(\alpha) = \pm 2$ for all α in the image of $i^*\colon \pi_1(\partial M) \to \pi_1(M)$,*
(iii) $\ker(\rho \circ i^*)$ *is the cyclic group generated by $[\gamma] \in \pi_1(\partial M)$.*

A slope is called regular *if it is not irregular.*

Remark 4.5. *In* [11]*, irregular slopes are called* exceptional slopes*. However, the term "exceptional slope" is commonly used to refer to a slope γ along which the Dehn filled manifold $M(\gamma)$ is not hyperbolic. The term irregular is used here to avoid any ambiguity.*

With these definitions, we are almost ready to state the Dehn surgery formula. But first we recall some useful notation for Dehn fillings along knot complements. For any choice of basis (α, β) for $H_1(\partial M;\mathbb{Z})$, there is a bijective correspondence between unoriented isotopy classes of simple closed curves in ∂M and elements in $\mathbb{Q} \cup \{\infty\}$ given by $\gamma \mapsto p/q$, where $\gamma = p\alpha + q\beta$. If M is the complement of a knot K in an integral homology sphere Σ, then the meridian $\mathscr{M}$ and longitude $\mathscr{L}$ of K provide a preferred basis for $H_1(\partial M;\mathbb{Z})$. We set $K(p/q) = M(\gamma)$, the Dehn filling along the curve $\gamma = p\mathscr{M} + q\mathscr{L}$. In this case, we call p/q the slope of γ and $K(p/q)$ the result of p/q Dehn surgery along the knot K in the homology 3-sphere Σ.

Definition 4.6. *A slope p/q is called* admissible *for K if*

(i) *p/q is a regular slope which is not a strict boundary slope.*
(ii) *no p'-th root of unity is a root of the Alexander polynomial of K, where $p' = p$ if p is odd and $p' = p/2$ if p is even.*

The next result is a restatement of Theorem 4.8 of [11], as corrected in [12].

Theorem 4.7. *Suppose K is a small knot in a homology 3-sphere Σ with complement M. Let $\{X_i\}$ be the collection of all one-dimensional components of the character variety $X(M)$ such that $r(X_i)$ is one-dimensional and such that $X_i \cap X^*(M)$ is nonempty. Define $\sigma\colon \mathbb{Z} \to \{0,1\}$ by $\sigma(p) = 0$ if p is even and $\sigma(p) = 1$ if p is odd.*

Then there exist integral weights $n_i > 0$ depending only on X_i and non-negative numbers E_0 and E_1 in $\frac{1}{2}\mathbb{Z}$ depending only on K such that for every admissible slope p/q, we have

$$\lambda_{SL_2(\mathbb{C})}(K(p/q)) = \sum_i n_i \|p\mathscr{M} + q\mathscr{L}\|_i - E_{\sigma(p)}.$$

In fact each character $\chi \in X^*(W_1) \cap X^*(W_2)$ contributes equally to each side of the equation in Theorem 4.7.

Proposition 4.8. *Suppose K is a small knot in a homology 3-sphere Σ with complement M. Let $\alpha = p\mathscr{M} + q\mathscr{L} \in H_1(M;\mathbb{Z})$ and suppose $\chi \in X^*(M)$ extends to give an isolated character in $X^*(K(p/q))$. Assume further that $\chi(\mathscr{M}) \neq \pm 2$ or $\chi(\mathscr{L}) \neq \pm 2$. Then the intersection multiplicity of $X^*(W_1)$ and $X^*(W_2)$ at χ is given by*

$$m_\chi = \tfrac{1}{2} \sum_i n_i \mu_{i,\chi},$$

where $\mu_{i,\chi}$ is the order of vanishing of $f_{i,\alpha}(\chi) = 0$. In particular, it follows that

$$m_\chi \geq \sum_{\{i \mid \chi \in X_i\}} n_i.$$

A proof of Proposition 4.8 is included in Section 6 for completeness.

Let $\pi^*\colon X(K(p/q)) \to X(M)$ be the injective map induced by the surjection $\pi\colon \pi_1 M \to \pi_1(K(p/q))$.

Corollary 4.9. *Let K be a small knot in an integral homology sphere with complement M. Suppose $\rho\colon \pi_1(K(p/q)) \to SL_2(\mathbb{C})$ is an irreducible representation such that $H^1(K(p/q); \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$. Let χ_ρ be the character of ρ, and suppose either $\chi_\rho(\mathscr{M}) \neq \pm 2$ or $\chi_\rho(\mathscr{L}) \neq \pm 2$. Then there is a unique curve X_i in $X(M)$ containing $\pi^*\chi_\rho$, and the associated weight of X_i is $n_i = 1$.*

Proof. Since $H^1(K(p/q); \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$, we know that the intersection multiplicity of $X^*(W_1)$ and $X^*(W_2)$ at χ_ρ is 1 by Theorem 2.1. Then the assertion follows immediately from the proposition. □

Thus, finding a character $\chi_\rho \in X^*(K(p/q))$ with $H^1(K(p/q); \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$ and $\chi_\rho(\mathscr{M}) \neq \pm 2$ or $\chi_\rho(\mathscr{L}) \neq \pm 2$ determines the weight n_i for the corresponding curve $X_i \subset X(M)$. This weight in turn yields information about the intersection multiplicities for all characters coming from X_i in the closed manifolds $K(a/b)$ for all regular, non-boundary slopes a/b. In particular, if every curve X_i in $X(M)$

contains a character which is the image of the character of a representation ρ with $\chi_\rho \in X^*(K(p/q))$ for some p/q and with $H^1(K(p/q); \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$, then one can determine $\lambda_{SL_2(\mathbb{C})}(K(a/b))$ for all admissible slopes a/b directly from information about the seminorms $\|\cdot\|_i$ and the numbers E_0, E_1. We apply this in the next section to manifolds obtained by Dehn surgery on twist knots.

5. Dehn surgeries on twist knots

Let K_ξ for $\xi \geq 1$ be the ξ-twist knot in S^3 depicted in Figure 1, with complement M_ξ, and let $K_\xi(p/q)$ denote the 3-manifold obtained by performing p/q Dehn surgery on K_ξ. In this section, we present formulas for $\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q))$ for all slopes p/q which are not strict boundary slopes. Note that K_ξ is hyperbolic for $\xi > 1$.

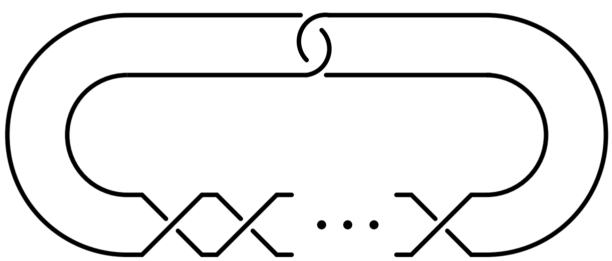

Fig. 1. The twist knot K_ξ with a clasp and ξ half twists.

By [7], all irreducible characters in $X(M_\xi)$ lie on a single curve in $X(M_\xi)$ which we denote X_ξ. We denote by $\|\cdot\|_\xi$ the seminorm on $H_1(\partial M_\xi; \mathbb{R})$ obtained from this curve. (Note that when $\xi > 1$, $\|\cdot\|_\xi$ is actually a norm, since K_ξ is hyperbolic.) Let n_ξ denote the weight of X_ξ described in Theorem 4.7.

Proposition 5.1. *Suppose p/q is an admissible slope for K_ξ. Then*

$$\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q)) = \|p\mathscr{M} + q\mathscr{L}\|_\xi - E_{\sigma(p)}.$$

Proof. By Theorem 4.7 above, since $X(M_\xi)$ contains a single curve X_ξ containing irreducible characters, we must show that $n_\xi = 1$.

It is well-known that K_ξ is small. In fact, if $\xi = 2k-1$ is odd, then

$$K_{2k-1}(1) = \Sigma(2, 3, 6k-1).$$

One can demonstrate this by identifying both manifolds with the one obtained by $1/k$ Dehn surgery on the left-handed trefoil using Kirby calculus. Similarly, if $\xi = 2k$ is even, then

$$K_{2k}(-1) = -\Sigma(2, 3, 6k+1).$$

By Theorem 2.3, each of these Seifert fibered spaces Σ admits irreducible representations $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$, and each such ρ satisfies $H^1(\Sigma; \mathfrak{sl}_2(\mathbb{C})_{Ad\,\rho}) = 0$. We show that each of these Seifert fibered spaces admits an irreducible representation ρ

with character $\chi_\rho \in X^*(\Sigma)$ satisfying $\chi_\rho(\mathscr{M}) \neq \pm 2$. Then Corollary 4.9 will imply that $n_\xi = 1$.

We use Casson's $SU(2)$ invariant to establish the existence of such characters. We have

$$-\lambda_{SU(2)}(K_{2k-1}(1)) = k = \lambda_{SU(2)}(K_{2k}(-1)).$$

Thus there exists an irreducible $SU(2)$ representation ρ of $\pi_1(\Sigma)$, since $k \neq 0$. But $\pi_1(\Sigma)$ is normally generated by the meridian $\mathscr{M}$, so $\rho(\mathscr{M}) \neq \pm I$ since ρ is irreducible. Since $\rho(\mathscr{M}) \in SU(2)$, it follows that $\chi_\rho(\mathscr{M}) \neq \pm 2$. □

Thus, $\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q))$ is completely determined by the seminorm $\|\cdot\|_\xi$ and the numbers E_0 and E_1 for admissible slopes. These will be computed in Propositions 5.5 and 5.6, but first, we investigate admissibility for slopes of twist knots.

The first lemma establishes regularity for all slopes of twist knots.

Lemma 5.2. *If K_ξ is a nontrivial twist knot, then all slopes are regular.*

Proof. We have the presentation $\pi_1(M_\xi) = \langle x, y \mid xw_\xi = w_\xi y\rangle$, where $w_\xi = (yx^{-1}y^{-1}x)^{\xi/2}$ if ξ is even, and $w_\xi = (yxy^{-1}x^{-1})^{(\xi-1)/2}yx$ if ξ is odd. Then as in section 7 of [8], the curve X_ξ in $X(M_\xi)$ can be parameterized by characters of representations $\rho\colon \pi_1(M_\xi) \to SL_2(\mathbb{C})$ with

$$\rho(x) = \begin{bmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{bmatrix} \quad \text{and} \quad \rho(y) = \begin{bmatrix} \mu & 0 \\ t_\xi & \mu^{-1} \end{bmatrix},$$

where t_ξ is chosen so that $\rho(xw_\xi) = \rho(w_\xi y)$. Henceforth we will write w for w_ξ and t for t_ξ whenever no confusion will arise by our doing so.

With this parameterization of X_ξ, we may identify the meridian $\mathscr{M}$ with x, so $\rho(\mathscr{M}) = \rho(x)$. If w^* denotes the word obtained by reversing w, then the longitude $\mathscr{L}$ is given by $\mathscr{L} = ww^*$ if ξ is even and by $\mathscr{L} = x^{-4}ww^*$ if ξ is odd.

Suppose p/q is an irregular slope. Then there exists an irreducible representation ρ of the knot complement with $\chi_\rho(\mathscr{M}) = \pm 2, \chi_\rho(\mathscr{L}) = \pm 2$, and $\rho(\mathscr{M}^p\mathscr{L}^q) = I$. We show that this cannot occur.

Since $\chi(\mathscr{M}) = \pm 2$, we know that $\mu = \mu^{-1} = \pm 1$. Suppose

$$\rho(w) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then

$$\rho(xw) = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \mu\alpha + \gamma & \mu\beta + \delta \\ \mu\gamma & \mu\delta \end{bmatrix}$$

and

$$\rho(wy) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \mu & 0 \\ t & \mu \end{bmatrix} = \begin{bmatrix} \mu\alpha + t\beta & \mu\beta \\ \mu\gamma + t\delta & \mu\delta \end{bmatrix}.$$

Note that $\rho(xw) = \rho(wy)$ implies that $\delta = 0$. It follows that $\gamma = -\beta^{-1}$ since $\rho(w) \in SL_2(\mathbb{C})$.

Now the presentation of $\pi_1(M_\xi)$ implies that $w_{\xi+2} = yx^{-1}y^{-1}xw_\xi$ if ξ is even and $w_{\xi+2} = yxy^{-1}x^{-1}w_\xi$ if ξ is odd. Then simple proofs by induction on ξ (treating

ξ odd and ξ even separately) show that

$$\rho(w^*) = \begin{bmatrix} 0 & \beta \\ -\beta^{-1} & \alpha \end{bmatrix}.$$

Hence

$$\rho(ww^*) = \begin{bmatrix} \alpha & \beta \\ -\beta^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ -\beta^{-1} & \alpha \end{bmatrix} = \begin{bmatrix} -1 & 2\alpha\beta \\ 0 & -1 \end{bmatrix}$$

and

$$\rho(x^{-4}ww^*) = \begin{bmatrix} \mu & -1 \\ 0 & \mu \end{bmatrix}^4 \begin{bmatrix} -1 & 2\alpha\beta \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2\alpha\beta + 4\mu^3 \\ 0 & -1 \end{bmatrix}.$$

Thus we have

$$\rho(\mathscr{M}) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix} \quad \text{and} \quad \rho(\mathscr{L}) = \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}$$

where $u = 2\alpha\beta$ if ξ is even and $u = 2\alpha\beta + 4\mu^3$ if ξ is odd. An easy computation shows that if $\chi(\mathscr{M}) = 2$, then

$$\rho(\mathscr{M}^p\mathscr{L}^q) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^p \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}^q = (-1)^q \begin{bmatrix} 1 & p - qu \\ 0 & 1 \end{bmatrix}. \tag{5.1}$$

On the other hand, if $\chi(\mathscr{M}) = -2$, then

$$\rho(\mathscr{M}^p\mathscr{L}^q) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^p \begin{bmatrix} -1 & u \\ 0 & -1 \end{bmatrix}^q = (-1)^{p+q} \begin{bmatrix} 1 & -p - qu \\ 0 & 1 \end{bmatrix}. \tag{5.2}$$

Claim. If u is rational, then it is an even integer.

Assume $u \in \mathbb{Q}$. Using that $u = 2\alpha\beta$ when ξ is even and that $u = 2\alpha\beta + 4\mu^3$ when ξ is odd, together with the fact that $\mu = \pm 1$, the assumption implies that $\alpha\beta \in \mathbb{Q}$. The claim will follow once we show $\alpha\beta \in \mathbb{Z}$.

Arguing by induction on ξ as before, it is not difficult to establish that for each $\xi \in \mathbb{N}$ and for $\mu = \pm 1$:

(i) $\alpha(t), \beta(t), \gamma(t)$, and $\delta(t)$ are all polynomials with integer coefficients.
(ii) $\delta(t)$ is a monic polynomial.
(iii) $\gamma(t) = t\beta(t)$.
(iv) If ξ is even, then $\alpha(t) - \delta(t) = (2 - t)\beta(t)$.
(v) If ξ is odd, then $\alpha(t) + \delta(t) = (2 + t)\beta(t)$.

Fix ξ. Recall that t_ξ is chosen so that $\rho(xw_\xi) = \rho(w_\xi y)$, which implies that $\delta(t_x i) = 0$. Assuming that $t_\xi \in \mathbb{Q}$, then it follows from the fact that t_ξ is the root of a monic polynomial with integer coefficients that $t_\xi \in \mathbb{Z}$, and then (i) implies that $\alpha(t_\xi)\beta(t_\xi) \in \mathbb{Z}$ as claimed. Thus the claim will follow once we show that $t_\xi \in \mathbb{Q}$.

Since $\gamma(t_\xi) = -\beta(t_\xi)^{-1}$, (iii) implies that $t_\xi\beta(t_\xi)^2 = -1$. Also $\alpha(t_\xi) = \alpha(t_\xi) - \delta(t_\xi) = \alpha(t_\xi) + \delta(t_\xi)$. Now if ξ is even, then (iv) shows that

$$\alpha(t_\xi)\beta(t_\xi) = (\alpha(t_\xi) - \delta(t_\xi))\beta(t_\xi) = (2 - t_\xi)\beta(t_\xi)^2 = 2\beta(t_\xi)^2 + 1.$$

Similarly, if ξ is odd, then (v) gives that

$$\alpha(t_\xi)\beta(t_\xi) = (\alpha(t_\xi) + \delta(t_\xi))\beta(t_\xi) = (2 + t_\xi)\beta(t_\xi)^2 = 2\beta(t_\xi)^2 - 1.$$

In either case, we can solve for $\beta(t_\xi)^2$ in terms of $\alpha(t_\xi)\beta(t_\xi)$. Since $\alpha(t_\xi)\beta(t_\xi) \in \mathbb{Q}$, we have $\beta(t_\xi)^2 \in \mathbb{Q}$. The equation $t_\xi \beta(t_\xi)^2 = -1$ then implies that $t_\xi \in \mathbb{Q}$, and this completes the proof of the claim.

By the claim, either $u \notin \mathbb{Q}$ or u is an even integer. Assume first that $\chi(\mathscr{M}) = 2$. If $u \notin \mathbb{Q}$, then $p \neq qu$ and $\rho(\mathscr{M}^p \mathscr{L}^q) \neq I$ by equation (5.1). On the other hand, if u is an even integer and $\rho(\mathscr{M}^p \mathscr{L}^q) = I$, then equation (5.1) implies that q is even and that $p = qu$. It follows that p is also even, which contradicts the assumption that p and q are relatively prime.

Assume now that $\chi(\mathscr{M}) = -2$. If $u \notin \mathbb{Q}$, then $p \neq -qu$ and $\rho(\mathscr{M}^p \mathscr{L}^q) \neq I$ by equation (5.2). On the other hand, if u is an even integer and $\rho(\mathscr{M}^p \mathscr{L}^q) = I$, then equation (5.2) implies that $p+q$ is even and that $p = -qu$. Hence p is even, as is q, again contradicting the assumption that p and q are relatively prime. This completes the proof of the lemma. $\square$

The next lemma discusses the roots of Alexander polynomials of twist knots.

Lemma 5.3. *The Alexander polynomial $\Delta_{K_\xi}(t)$ of K_ξ has a root which is a k-th root of unity if and only if $\xi = 1$. The roots of $\Delta_{K_1}(t)$ are 6-th roots of unity.*

Proof. The Alexander polynomial of K_ξ is given by

$$2\Delta_{K_\xi}(t) = \begin{cases} \xi t^2 - 2(\xi+1)t + \xi & \text{if } \xi \text{ is even,} \\ (\xi+1)t^2 - 2\xi t + (\xi+1) & \text{if } \xi \text{ is odd.} \end{cases}$$

If $\xi = 1$, then $\Delta_{K_1}(t) = t^2 - t + 1$ with roots $e^{\pm \pi i/3}$, which are 6-th roots of unity. We consider the remaining cases separately.

Case 1: $\xi > 1$ is even.

Then the roots of $\Delta_{K_\xi}(t)$ are real and equal to $(\xi + 1 \pm \sqrt{2\xi+1})/\xi$, neither of which has absolute value equal to 1.

Case 2: $\xi > 1$ is odd.

Then the roots of $\Delta_{K_\xi}(t)$ are not real. Suppose $\Delta_{K_\xi}(t)$ has a root of the form $\alpha = e^{2\pi i j/k}$. Then $2\Delta_{K_\xi}(t) = (\xi+1)(t-\alpha)(t-\bar{\alpha})$. Thus, $2\Delta_{K_\xi}(t)$ divides the polynomial $(\xi+1)(t^k - 1)$. It follows that $2\Delta_{K_\xi}(t)/(\xi+1) = t^2 - [2\xi/(\xi+1)]t + 1$ divides $t^k - 1$. We will show this leads to a contradiction.

Let $t^2 - \beta t + 1$ be a quadratic polynomial which divides $t^k - 1$. We claim that if β is rational, then it is integral. Proving this claim will complete the proof of Case 2, since $2\xi/(\xi+1)$ is rational but not integral for $\xi > 1$.

To verify the claim, we write $t^k - 1 = (t^2 - \beta t + 1) \cdot Q(t)$ and solve for the coefficients of $Q(t) = \sum_{i=0}^{k-2} q_i t^{k-i-2}$. We find that $q_0 = 1$, $q_1 = \beta$, and $q_i = \beta q_{i-1} - q_{i-2}$ for $i \geq 2$. Thus $q_i = q_i(\beta)$ is a monic polynomial of degree i in β with integer coefficients. Moreover $q_{k-2} = -1$, so β is a root of the monic polynomial $q_{k-2}(x) + 1$. But any rational root of a monic polynomial with integer coefficients is integral. This establishes the claim and thereby the lemma. $\square$

These lemmas allow us to precisely identify the admissible slopes for K_ξ:

Proposition 5.4. *Every slope is admissible for K_1 except the strict boundary slope and slopes of the form p/q where p is a nonzero multiple of 12. For $\xi > 1$, every slope is admissible for K_ξ except the strict boundary slopes.*

The only strict boundary slope for K_1 is 6.

The strict boundary slopes for K_2 are -4 and 4.

The strict boundary slopes for $K_\xi, \xi \geq 3$ odd, are $0, 4,$ and $2\xi + 4$.

The strict boundary slopes for $K_\xi, \xi \geq 4$ even, are $-4, 0,$ and 2ξ.

Proof. The first two claims follow from Lemmas 5.2 and 5.3, so all that remains is to verify the lists of strict boundary slopes. The boundary slopes for K_ξ are determined in [15] to be 0 and 6 if $\xi = 1$; $-4, 0,$ and 2ξ if $\xi \geq 2$ is even; and $0, 4,$ and $2\xi + 4$ if $\xi \geq 3$ is odd. But the trefoil K_1 and the figure eight knot K_2 are fibered knots with 0 the slope of the fiber. Thus, the only strict boundary slope for the trefoil is 6, and the strict boundary slopes of the knots K_ξ for $\xi > 1$ are as given above. □

In light of Proposition 5.1, we find that computing the seminorm $\|\cdot\|_\xi$ and E_σ, $\sigma = 0, 1$, for K_ξ completely determines $\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q))$ for all admissible slopes.

We compute the Culler-Shalen norm $\|\cdot\|_{CS}$ for K_ξ, $\xi > 1$, which determines $\|\cdot\|_\xi$ since $\|\cdot\|_\xi = (1/4)\|\cdot\|_{CS}$. These norms were computed previously by Boyer, Mattman, Zhang in [2]. An alternative method for computing these norms was developed by Ohtsuki in [18].

Proposition 5.5. *The Culler-Shalen norm for K_ξ, $\xi > 1$, is given by*

$$\|p\mathscr{M} + q\mathscr{L}\|_{CS} = \begin{cases} \xi|4q+p| + (\xi-2)|p| + 2|2\xi q - p| & \text{if } \xi \text{ is even,} \\ (\xi-1)|4q-p| + (\xi-1)|p| + 2|(2\xi+4)q - p| & \text{if } \xi \text{ is odd.} \end{cases}$$

Proof. Recall that K_ξ is hyperbolic, so $\|\cdot\|$ is a norm on $X(M_\xi)$. In the proof of Proposition 8.6 of [5], it is shown that

$$\|p\mathscr{M} + q\mathscr{L}\|_{CS} = \sum a_i|u_i q - v_i p|,$$

where the sum is taken over all boundary slopes u_i/v_i and where $a_i \in 2\mathbb{Z}$ is non-negative.

Again using the enumeration of the boundary slopes for K_ξ in [15], we have

$$\|p\mathscr{M} + q\mathscr{L}\|_{CS} = \begin{cases} a_1|4q+p| + a_2|p| + a_3|2\xi q - p| & \text{if } \xi \text{ is even,} \\ b_1|4q-p| + b_2|p| + b_3|(2\xi+4)q - p| & \text{if } \xi \text{ is odd.} \end{cases}$$

In [2], the authors compute $\|\cdot\|_{CS}$ for the slopes $\mathscr{M}$, $-\mathscr{M} + \mathscr{L}$, and $-2\mathscr{M} + \mathscr{L}$ for each K_ξ with ξ even and for the mirror image of each K_ξ with ξ odd. Recalling that p/q Dehn surgery on a knot is equivalent to $-p/q$ Dehn surgery on its mirror image, we may use these values to determine $\|\cdot\|_{CS}$ as follows:

If ξ is even we obtain the equations

$$\begin{aligned} 2\xi &= \|\mathscr{M}\|_{CS} = a_1 + a_2 + a_3 \\ 8\xi &= \|-\mathscr{M}+\mathscr{L}\|_{CS} = 3a_1 + a_2 + (2\xi+1)a_3 \\ 8\xi &= \|-2\mathscr{M}+\mathscr{L}\|_{CS} = 2a_1 + 2a_2 + (2\xi+2)a_3 \end{aligned}$$

with solution $a_1 = \xi$, $a_2 = \xi - 2$, and $a_3 = 2$ easily determined by linear algebra.

If ξ is odd, we obtain the equations

$$\begin{aligned} 2\xi &= \|-\mathscr{M}\|_{CS} = b_1 + b_2 + b_3 \\ 8\xi+2 &= \|\mathscr{M}+\mathscr{L}\|_{CS} = 3b_1 + b_2 + (2\xi+3)b_3 \\ 8\xi &= \|2\mathscr{M}+\mathscr{L}\|_{CS} = 2b_1 + 2b_2 + (2\xi+2)b_3 \end{aligned}$$

with solution $b_1 = \xi - 1 = b_2$ and $b_3 = 2$. □

We next compute the correction terms E_0 and E_1 for the twist knots.

Proposition 5.6. *For the twist knots K_ξ, $\xi \geq 1$, we have $E_0 = 0$ and $E_1 = \xi/2$.*

Proof. We begin with E_1. Note that the meridian $\mathscr{M}$ is a regular slope and is not a boundary slope for K_ξ for any ξ. Therefore by Theorem 4.7 we have

$$\begin{aligned} \lambda_{SL_2(\mathbb{C})}(K_\xi(1/0)) &= \|\mathscr{M}\|_\xi - E_1 \\ &= 1/4\|\mathscr{M}\|_{CS} - E_1. \end{aligned}$$

But $K_\xi(1/0)$ is simply S^3, so $E_1 = 1/4\|\mathscr{M}\|_{CS} = \xi/2$ by Proposition 5.5.

We now turn to the computation of E_0. By Theorem 4.7, we know that

$$E_0 = \|2\mathscr{M}+\mathscr{L}\|_\xi - \lambda_{SL_2(\mathbb{C})}(K_\xi(2/1)).$$

Moreover, by Proposition 4.8 and Corollary 4.9, we know that all characters χ with $\chi(\mathscr{M}) \neq \pm 2$ or $\chi(\mathscr{L}) \neq \pm 2$ contribute equally to both $\|2\mathscr{M}+\mathscr{L}\|_\xi$ and $\lambda_{SL_2(\mathbb{C})}(K_\xi(2/1))$. Thus, E_0 measures the difference in the contributions of characters χ with $\chi(\mathscr{M}) = \pm 2$ and $\chi(\mathscr{L}) = \pm 2$ to the quantities $\|2\mathscr{M}+\mathscr{L}\|_\xi$ and $\lambda_{SL_2(\mathbb{C})}(K_\xi(2/1))$. However, by the proof of Lemma 5.2, for any character χ with $\chi(\mathscr{M}) = \pm 2$ we have $\chi(2\mathscr{M}+\mathscr{L}) = -2$. Thus, there are no characters χ with $\chi(\mathscr{M}) = \pm 2$ and $\chi(\mathscr{L}) = \pm 2$ satisfying $f_{2\mathscr{M}+\mathscr{L}}(\chi) = 0$. By Proposition 4.2 of [11], it follows that there are no characters χ with $\chi(\mathscr{M}) = \pm 2$ and $\chi(\mathscr{L}) = \pm 2$ which contribute to $\lambda_{SL_2(\mathbb{C})}(K_\xi(2/1))$. Therefore all characters χ with $\chi(\mathscr{M}) = \pm 2$ and $\chi(\mathscr{L}) = \pm 2$ contribute 0 to both $\|2\mathscr{M}+\mathscr{L}\|_\xi$ and $\lambda_{SL_2(\mathbb{C})}(K_\xi(2/1))$, and $E_0 = 0$. □

Summarizing, we have the following formulas for $\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q))$:

Theorem 5.7. *Suppose $\xi > 1$ and p/q is not a strict boundary slope for K_ξ. Then the $SL_2(\mathbb{C})$ Casson invariant of $K_\xi(p/q)$ is as follows:*

If ξ is even, then

$$\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q)) = \begin{cases} \frac{1}{4}(\xi|4q+p| + (\xi-2)|p| + 2|2\xi q - p|) & \text{if } p \text{ is even,} \\ \frac{1}{4}(\xi|4q+p| + (\xi-2)|p| + 2|2\xi q - p| - 2\xi) & \text{if } p \text{ is odd.} \end{cases}$$

If ξ is odd, then

$$\lambda_{SL_2(\mathbb{C})}(K_\xi(p/q)) = \begin{cases} \frac{1}{4}\left((\xi-1)(|4q-p|+|p|)+2|2\xi q+4q-p|\right) & \text{if } p \text{ is even,} \\ \frac{1}{4}\left((\xi-1)(|4q-p|+|p|)+2|2\xi q+4q-p|-2\xi\right) & \text{if } p \text{ is odd.} \end{cases}$$

The next two results treat the trefoil K_1 as a special case.

Proposition 5.8. *The Culler-Shalen norm for the right hand trefoil K_1 is given by*

$$\|p\mathscr{M}+q\mathscr{L}\|_{CS} = 2|6q-p|$$

Proof. Proposition 5.4 shows that 6 is the only strict boundary slope for K_1. Using the parameterization of $X^*(M_1)$ described in the proof of Lemma 5.2, it is easy to check that $\chi(\mathscr{M}^6\mathscr{L}) = -2$ for every $\chi \in X^*(M_1)$. Therefore by Proposition 5.4 of [4], $\|\cdot\|_{CS}$ is an indefinite seminorm which can be written as $\|p\mathscr{M}+q\mathscr{L}\|_{CS} = m|6q-p|$ for some $m \in \mathbb{Z}$. Theorem 2.3 gives that $\lambda_{SL_2(\mathbb{C})}(K_1(1)) = \lambda_{SL_2(\mathbb{C})}(\Sigma(2,3,5)) = 2$, and Proposition 5.6 implies $E_1 = 1/2$. Hence $\|\mathscr{M}+\mathscr{L}\|_1 = 5/2$ and $\|\mathscr{M}+\mathscr{L}\|_{CS} = 10$. Thus $m = 2$, and $\|\cdot\|_{CS}$ is as stated in the proposition. □

Theorem 5.9. *The $SL_2(\mathbb{C})$-Casson invariant for the right hand trefoil K_1 is given by*

$$\lambda_{SL_2(\mathbb{C})}(K_1(p/q)) = \begin{cases} \frac{1}{2}|6q-p| - \frac{1}{2} & \text{if } p \text{ is odd,} \\ \frac{1}{2}|6q-p| & \text{if } p \text{ is even and not a multiple of } 12, \\ \frac{1}{2}|6q-p| - 2 & \text{if } p \text{ is a multiple of } 12. \end{cases}$$

Proof. The theorem follows from Propositions 5.1, 5.6, and 5.8, provided p is not a multiple of 12. Of course, one must check the strict boundary slope $p/q = 6$ by hand. Since $K_1(6) = L(2,1)\#L(3,1)$, its fundamental group $\pi_1(K_1(6)) = \mathbb{Z}_2 * \mathbb{Z}_3$ does not admit irreducible $SL_2(\mathbb{C})$ representations and we see that $\lambda_{SL_2(\mathbb{C})}(K_1(6)) = 0$, in agreement with our formula. In what follows, we present the argument in the case $p \neq 0$ is a multiple of 12.

By Proposition 4.8, for any character χ in $X^*(K(p/q))$ such that either $\chi(\mathscr{M}) \neq \pm 2$ or $\chi(\mathscr{L}) \neq \pm 2$, the intersection multiplicity of $X^*(W_1)$ and $X^*(W_2)$ at χ is $1/2$ the order of vanishing of $f_{p\mathscr{M}+q\mathscr{L}}$ at χ, since $n_1 = 1$.

Now parameterizing X_1 as in the proof of Lemma 5.2, we see by the argument used in that proof that for any representation $\rho\colon \pi_1(M) \to SL_2(\mathbb{C})$ with character χ_ρ in $X^*(K_1(p/q))$ such that $\chi_\rho(\mathscr{M}) = \pm 2$, the matrix $\rho(\mathscr{M}^p)$ is upper triangular with diagonal entries equal to 1, and the matrix $\rho(\mathscr{L}^q)$ is upper triangular with diagonal entries equal to -1, since p is a multiple of 12 and q is odd. Thus, no characters χ with $\chi(\mathscr{M}) = \pm 2$ and $\chi(\mathscr{L}) = \pm 2$ contribute to $\lambda_{SL_2(\mathbb{C})}(K_1(p/q))$. Therefore $\lambda_{SL_2(\mathbb{C})}(K_1(p/q))$ is $1/2$ the sum of the orders of the zeros of $f_{p\mathscr{M}+q\mathscr{L}}$ at characters χ in $X^*(K_1(p/q))$.

We next observe that X_1 contains the characters of exactly two conjugacy classes of reducible representations: namely, the representations ρ_1 and ρ_2 where

$$\rho_1(x) = \begin{bmatrix} e^{\pi i/6} & 1 \\ 0 & e^{-\pi i/6} \end{bmatrix} \quad \text{and} \quad \rho_1(y) = \begin{bmatrix} e^{\pi i/6} & 0 \\ 0 & e^{-\pi i/6} \end{bmatrix}$$

and

$$\rho_2(x) = \begin{bmatrix} e^{5\pi i/6} & 1 \\ 0 & e^{-5\pi i/6} \end{bmatrix} \quad \text{and} \quad \rho_2(y) = \begin{bmatrix} e^{5\pi i/6} & 0 \\ 0 & e^{-5\pi i/6} \end{bmatrix}.$$

The order of vanishing of $f_{p\mathscr{M}+q\mathscr{L}}$ at each of the associated characters is 2 (which can be seen using an argument analogous to that used to prove Proposition 4.8). Therefore

$$\lambda_{SL_2(\mathbb{C})}(K_1(p/q)) = \tfrac{1}{2}\deg(f_{p\mathscr{M}+q\mathscr{L}}) - 2 = \tfrac{1}{4}\|p\mathscr{M}+q\mathscr{L}\|_{CS} - 2 = \tfrac{1}{2}|6q-p| - 2.$$

□

Theorems 2.3 and 5.7 can be used to deduce the knot invariant $\lambda'_{SL_2(\mathbb{C})}(K)$ defined in [11] for torus and twist knots.

Corollary 5.10. *(i) If $T_{p,q}$ is the p,q torus knot, then*

$$\lambda'_{SL_2(\mathbb{C})}(T_{p,q}) = \tfrac{1}{4}pq(p-1)(q-1).$$

(ii) If K_ξ is the ξ-twist knot, then

$$\lambda'_{SL_2(\mathbb{C})}(K_\xi) = \begin{cases} 2\xi & \text{if } \xi \text{ is even,} \\ 2\xi+1 & \text{if } \xi \text{ is odd.} \end{cases}$$

Proof. For any small knot K in an integral homology sphere, the invariant $\lambda'_{SL_2(\mathbb{C})}(K)$ is defined by the equation

$$\lambda'_{SL_2(\mathbb{C})}(K) = \lambda_{SL_2(\mathbb{C})}\left(K\left(\tfrac{1}{n+1}\right)\right) - \lambda_{SL_2(\mathbb{C})}\left(K\left(\tfrac{1}{n}\right)\right)$$

for n large. To prove part (i), we use the well-known fact that $1/n$ Dehn surgery on $T_{p,q}$ gives the Brieskorn sphere $\Sigma(p,q,pqn-1)$, and Theorem 2.3 implies that

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,pqn-1)) = \tfrac{1}{4}(p-1)(q-1)(pqn-2).$$

Part (ii) follows similarly from Theorems 5.7 and 5.9. □

It is an important problem to compute $\lambda_{SL_2(\mathbb{C})}(K(p/q))$ for surgeries on a broader collection of knots, say for 2-bridge knots and/or alternating knots. This data will help determine the relationship between $\lambda'_{SL_2(\mathbb{C})}(K)$ and classical knot invariants such as the Alexander polynomial and signature.

6. Proof of Proposition 4.8

This section is devoted to the proof of Proposition 4.8. Essentially, we review the proof of Theorem 4.8 of [11], explaining the correction of [12].

We begin with some definitions.

Let $\Delta \subset R(\partial M)$ be the subvariety of diagonal representations, and let $t_\Delta\colon \Delta \to X(\partial M)$ be the restriction to Δ of the canonical surjection $t\colon R(\partial M) \to X(\partial M)$. Then t_Δ is easily seen to be surjective. Let $r\colon X(M) \to X(\partial M)$ be the restriction map. Finally, for each component X_i of $X(M)$ such that $r(X_i)$ is one–dimensional, let D_i be the curve $t_\Delta^{-1}(\overline{r(X_i)}) \subset \Delta$.

Note that D_i is a branched double cover of the closure of $r(X_i)$. We have

$$\overline{r(X_i)} = t_\Delta(D_i) = \frac{D_i}{(a,b) \sim (a^{-1}, b^{-1})}$$

where $(a,b) \in \mathbb{C}^* \times \mathbb{C}^* \cong \Delta$.

Proof of Proposition 4.8. Note that by Propositions 4.1 and 4.2 of [11], since $\chi(\mathscr{M}) \neq \pm 2$ or $\chi(\mathscr{L}) \neq \pm 2$, we have $\chi \in X^*(W_1) \cap X^*(W_2)$ if and only if χ is a zero of $f_{p\mathscr{M}+q\mathscr{L}}$. By Proposition 4.3 of [11], the contribution of $r(\chi)$ to $\lambda_{SL_2(\mathbb{C})}(K(p/q))$ is

$$\sum_{\{i \mid \chi \in X_i\}} n_i d_i \phi_{i,\chi},$$

where n_i is a positive integer depending only on X_i, where d_i is the degree of the map $r|_{X_i} \colon X_i \to r(X_i)$, and where $\phi_{i,\chi}$ is the intersection number of $r(X_i)$ and $t_\Delta(\{(x,y) \mid x^p y^q = 1\})$ at $r(\chi)$. Equivalently, $\phi_{i,\chi}$ is $1/2$ the intersection multiplicity of D_i and the curve $\{(x,y) \mid x^p y^q = 1\}$ in $\mathbb{C}^* \times \mathbb{C}^*$, since $r(\chi)$ has two inverse images in $t_\Delta^{-1}(r(X_i)) = D_i$. Thus, the contribution of χ to $\lambda_{SL_2(\mathbb{C})}(K(p/q))$ is

$$\sum_{\{X_i \mid \chi \in X_i\}} \frac{n_i d_i \phi_{i,\chi}}{|r^{-1}(r(\chi))|},$$

where $|r^{-1}(r(\chi))|$ denotes the number of elements in the set $r^{-1}(r(\chi))$.

Our task is to show that $\mu_{i,\chi} = \frac{2 d_i \phi_{i,\chi}}{|r^{-1}(r(\chi))|}$.

For convenience, set $\alpha = p\mathscr{M} + q\mathscr{L}$. Let $g_{i,\alpha} \colon r(X_i) \to \mathbb{C}$ be the map taking an element $r(\chi) \in r(X_i)$ to $\chi(\mathscr{M}^p \mathscr{L}^q)$. Then $f_{i,\alpha}$ factors as $g_{i,\alpha} \circ r$. Thus,

$$\mu_{i,\chi} = \frac{d_i}{|r^{-1}(r(\chi))|} \cdot (\text{ order of vanishing of } g_{i,\alpha} \text{ at } r(\chi)).$$

Therefore, to prove the first assertion of the proposition, we must show that the order of the vanishing of $g_{i,\alpha}$ at $r(\chi)$ is the intersection multiplicity of D_i and the curve $\{(x,y) \mid x^p y^q = 1\}$ in $\mathbb{C}^* \times \mathbb{C}^*$.

This argument is a pointwise version of Proposition 6.6, Corollary 6.7, and Proposition 8.5 of [5].

Consider the commutative diagram:

$$\begin{tikzcd}
D_i \subset \mathbb{C}^* \times \mathbb{C}^* \arrow[r, "t_\Delta"] \arrow[d, "\Psi_{p,q}"'] & r(X_i) \arrow[d, "g_{i,\alpha}"] \\
\mathbb{C}^* \arrow[r, "z \mapsto z + z^{-1} - 2"] & \mathbb{C}
\end{tikzcd}$$

Here, $\Psi_{p,q} \colon \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ is the map $(x,y) \mapsto x^p y^q$. Observe that the two horizontal maps in this diagram have degree two.

We see that $r(\chi)$ is a zero of $g_{i,p\mathscr{M}+q\mathscr{L}}$ if and only if $(x,y) \in t_\Delta^{-1}(r(\chi))$ satisfies $\Psi_{p,q}(x,y) = 1$, and if so, then the orders of vanishing coincide. But the solutions to $\Psi_{p,q}(x,y) = 1$ are precisely the points in the intersection $D_i \cap \{(x,y) \mid x^p y^q = 1\}$,

and the order of vanishing equals the intersection multiplicity of D_i and $\{(x,y) \mid x^p y^q = 1\}$ in $\mathbb{C}^* \times \mathbb{C}^*$.

This proves the first assertion in the proposition. The final statement follows from the fact that $\mu_{i,\chi}/2$ is integral. To see this, note that $\frac{d_i}{|r^{-1}(r(\chi))|}$ is integral, and since $r(\chi)$ has two preimages in D_i, it follows that $\phi_{i,\chi}$ integral. $\square$

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